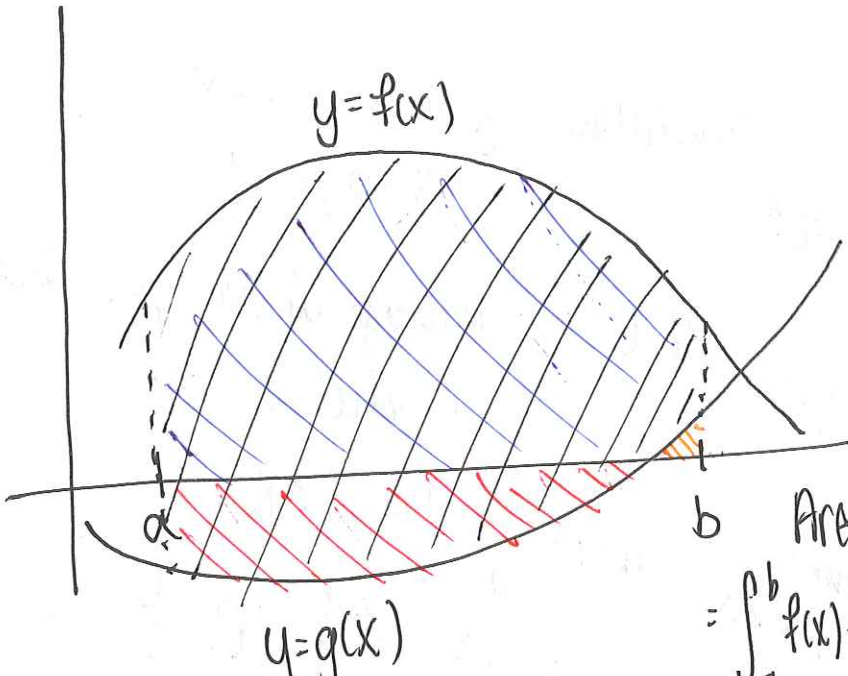
Feb. 28, 2014

Areas Between Curves

We want the area of the following region:



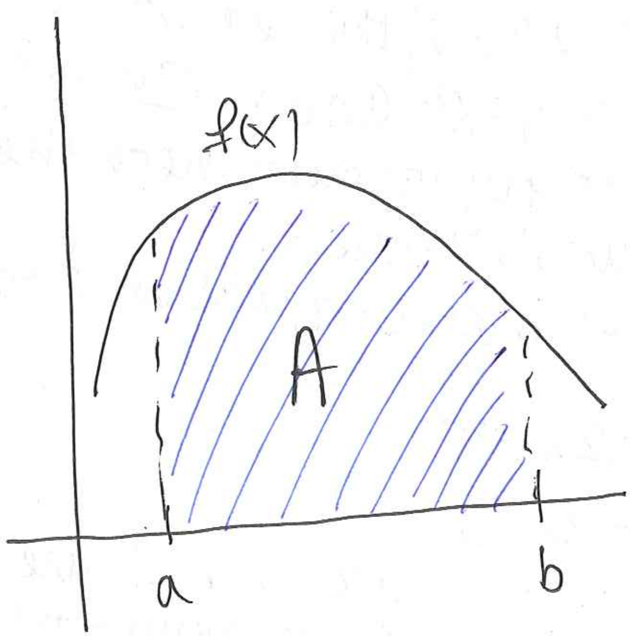
$$\text{Area} = A + B - C$$

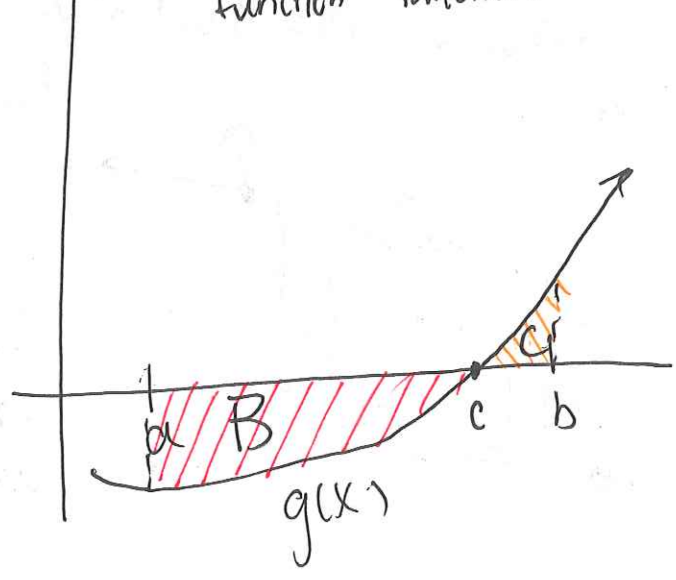
$$= \int_a^b f(x) + \int_a^c g(x)\,dx - \int_c^b g(x)\,dx$$

$$= \int_a^b f(x) - \int_a^b g(x)\,dx$$

$$= \int_a^b \underbrace{f(x)}_{\text{top function}} - \underbrace{g(x)}_{\text{bottom function}}\,dx$$

Consider f(x) and g(x) separately:

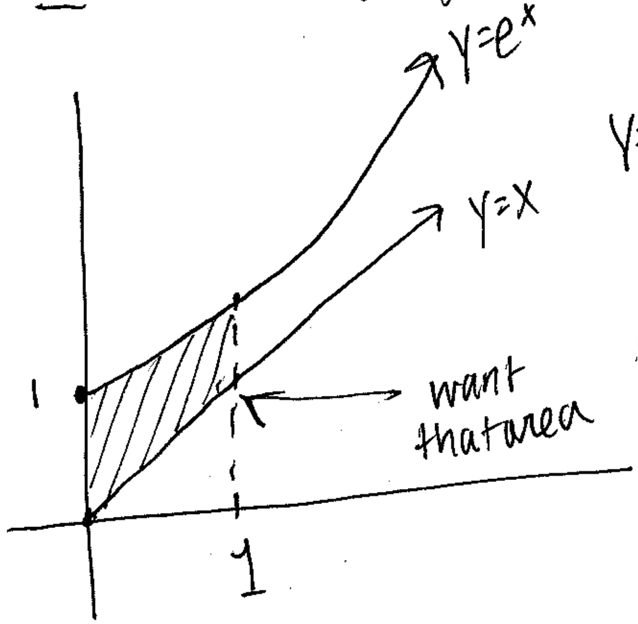




Area between curves $f(x)$, $g(x)$ and bounded by lines $x=a$ and $x=b$ when $f(x) \geq g(x)$ for all x in $[a,b]$ is

$$A = \int_a^b (f(x) - g(x))\, dx$$

Ex: (1) find area of region bounded by $y=e^x$, $y=x$, $x=0$, $x=1$



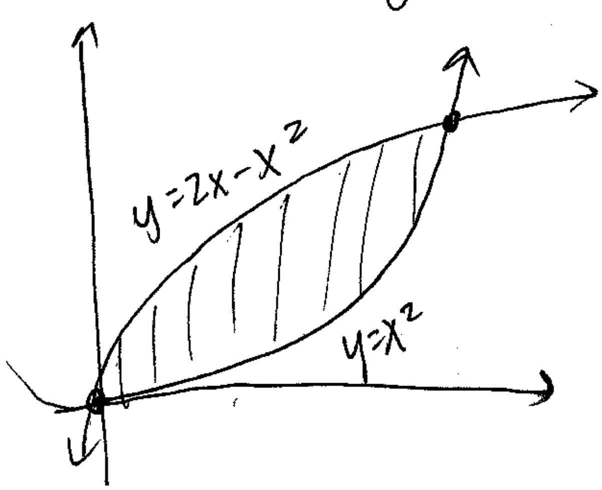
$y=e^x$ is always above $y=x$ between 0 and 1 so:

$$\text{Area} = \int_0^1 e^x - x \, dx$$

$$= e^x - \frac{x^2}{2} \Big|_0^1 = e - \frac{1}{2} - 1$$

$$= \boxed{e - 3/2}$$

(2) (Enclosed area) Find the area of the region enclosed by the parabolas $y=x^2$ and $y=2x-x^2$.



Need to figure out where the curves intersect.

- set them equal and solve!

$$x^2 = 2x - x^2$$

$$2x^2 - 2x = 0$$

$$2x(x-1) = 0 \implies x=0,\ x=1$$ are the only two locations the curves cross over.

$y = 2x - x^2$ is always above $y = x^2$ on the interval from $x = 0$ to $x = 1$:

$$\text{Area} = \int_0^1 2x - x^2 - x^2 \, dx = \int_0^1 2x - 2x^2 \, dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_0^1 = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

EX: Find the area of the region bounded by two curves $y = x^3 - 9x$ and $y = 9 - x^2$.

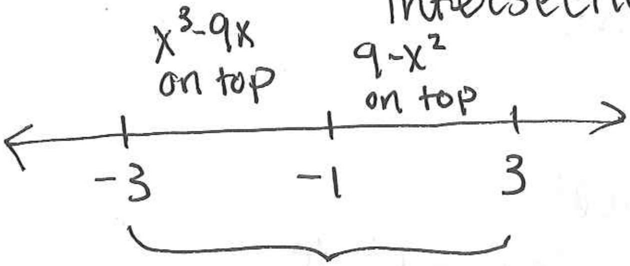
Find intersection points:

$$x^3 - 9x = 9 - x^2$$

$$x^3 + x^2 - 9x - 9 = 0$$

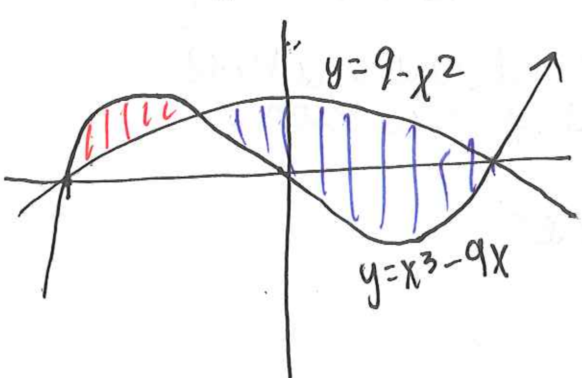
$$(x+1)(x-3)(x+3) = 0$$

Intersections at $-1, 3, -3$.



enclosed area is in this region

Check this is right w/ a graph



Check which curve is on top from $[-3, -1]$:

$$(-2)^3 - 9(-2) = -8 + 18 = 10 \quad \text{larger}$$

$$9 - (-2)^2 = 5$$

Check which curve is on top from $[-1, 3]$

$$0^3 - 9(0) = 0$$

$$9 - (0)^2 = 9 \quad \text{larger}$$

Calculate area:

$$\int_{-3}^{-1} (x^3-9x-(9-x^2))\,dx + \int_{-1}^{3} 9-x^2-(x^3-9x)\,dx$$

$$= \int_{-3}^{-1} x^3+x^2-9x-9\,dx + \int_{-1}^{3} -x^3-x^2+9x-9\,dx$$

$$= \left(\frac{x^4}{4}+\frac{x^3}{3}-\frac{9}{2}x^2-9x\right)\Bigg|_{-3}^{-1} + \left(-\frac{x^4}{4}-\frac{x^3}{3}+\frac{9}{2}x^2-9x\right)\Bigg|_{-1}^{3}$$

$$= \frac{148}{3}$$

Rule: the area between the curves $y=f(x)$ and $y=g(x)$ between $x=a$ and $x=b$ is

$$A=\int_a^b |f(x)-g(x)|\,dx$$

Process

(1) find where curves intersect

(2) between intersection pts, test a pt to determine if f or g is larger

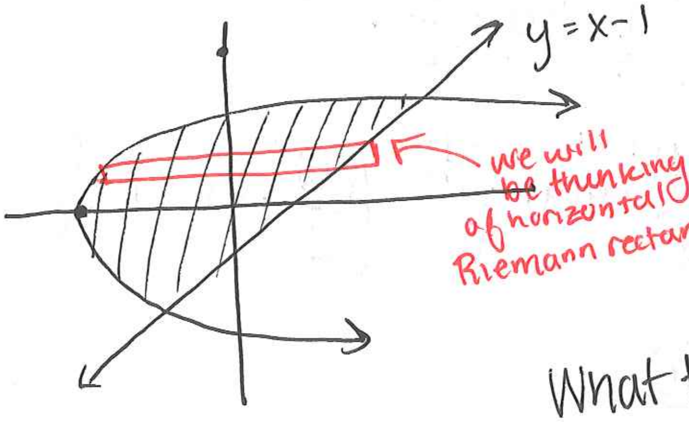
(3) integrate $f(x)-g(x)$ over intervals where $f(x)$ is larger

$g(x)-f(x)$ over intervals where $g(x)$ is larger

(4) add the integrals together.

A Trick:

Example: Find area enclosed by $y=x-1$ and $y^2=2x+6$.



~~f~~finding area in the way we have been would be very difficult (not impossible, but difficult).

What to do – think of your functions in terms of y, rather than x.

Observe:

$$y=x-1 \Longrightarrow x=y+1$$

$$y^2=2x+6 \Longrightarrow x=\frac{y^2}{2}-3$$

Everything the same as before, but now y is playing role of x.

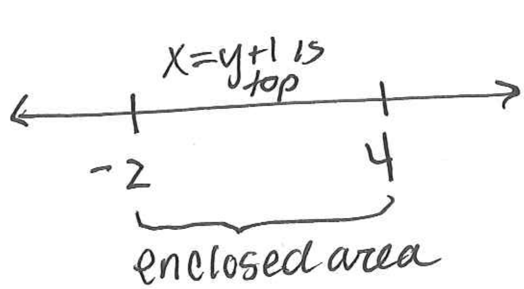
Find intersection points:

$$y+1=\frac{y^2}{2}-3$$

$$0=\frac{y^2}{2}-y-4$$

$$0=y^2-2y-8=(y-4)(y+2)$$

$y=4,\ y=-2$ are where curves intersect.



test at $y=0$:

$0+1=1 \longrightarrow x=y+1$ is on top

$\frac{0^2}{2}-3=-3$

Area: $\int_{-2}^{4} y+1-\left(\frac{y^2}{2}-3\right) dy$

$$= \int_{-2}^{4} y+4-\frac{y^2}{2}\, dy = \left.\frac{y^2}{2}+4y-\frac{y^3}{6}\right|_{-2}^{4}$$

$$= \frac{(4)^2}{2}+4(4)-\frac{(4)^3}{6}-\left(\frac{(-2)^2}{2}+4(-2)-\frac{(-2)^3}{6}\right)$$

$$= \boxed{18}$$